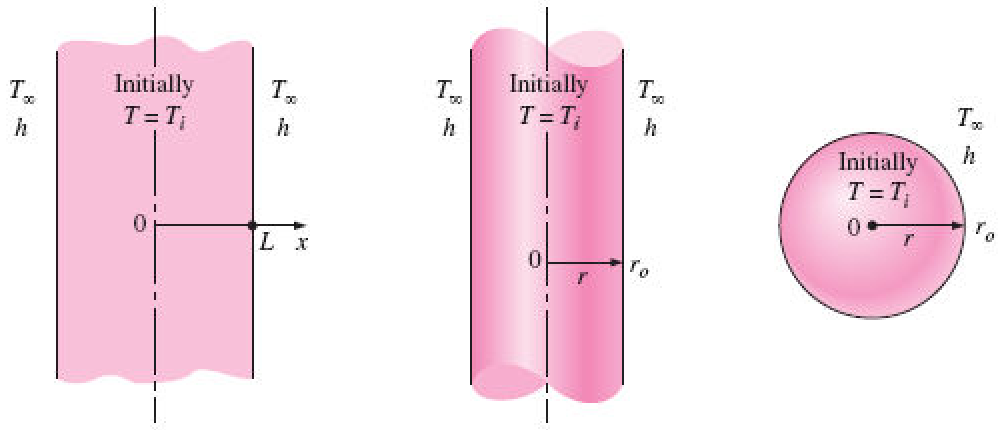
CHAPTER 2

THEORETICAL STUDY

2.1 Transient Heat Conduction in Large Plane Walls, Long Cylinders and Solid Spheres

In general, temperature within a body changes from point to point as well as with time. Therefore the problems can be considered as three dimensional and time dependent. In one dimensional problem such as those associated with a large plane wall, a long cylinder and a sphere the variation of temperature has been considered to be only in one of the dimensions and time dependent.



(a) A large plane wall (b) A long cylinder (c) A solid sphere

Figure 2.1: Schematic figures of the simple geometries in which heat transfer is one dimensional. [12]

It has been consider a plane wall of thickness 2L, a long cylinder of radius r_o, and a sphere of radius r_o, initially at a uniform temperature T_i, as shown in Figure 2.1 at time t = 0. Each geometry has been placed in a large medium that is at a constant temperature T_∞ and kept in that medium for $t > 0$. Heat transfer takes place between these bodies and their environments by convection with a uniform and constant heat transfer coefficient. It has been noted that all three cases possess geometric and thermal symmetry: the plane wall is symmetric about its centre plane, the cylinder is symmetric about its centreline and the sphere is symmetric about its centre point. Radiation heat transfer has been neglected between these bodies and their surrounding surfaces.

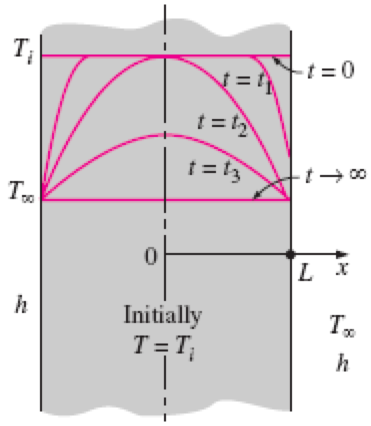


Figure 2.2: Transient temperature profiles in a plane wall exposed to convection from its surfaces for $T_i > T_\infty$

The variation of the temperature profile with time in the plane wall is illustrated in Figure 2.2. When the wall is first exposed to the surrounding medium at T_∞ < Ti at t = 0, the entire wall is at its initial temperature T_i. But the wall temperature at and near the surfaces starts to drop as a result of heat transfer from the wall to the surrounding medium. This creates a temperature gradient in the wall and initiates heat conduction from the inner parts of the wall toward its outer surfaces. It has been noted that the temperature at the centre of the wall remains at Ti until $t = t_2$ and that the temperature profile within the wall remains symmetric at all times about the centre plane. The temperature profile gets flatter and flatter as time passes as a result of heat transfer and eventually becomes uniform at $T = T_\infty$. That is, the wall reaches thermal equilibrium with its surroundings. At that point, the heat transfer stops since there is no longer a temperature difference. Similar discussions can be given for the long cylinder or sphere.

The formulation of the problems for the determination of the one dimensional transient temperature distribution T(x, t) in a wall results in a partial differential equation, which can be solved using the separation of variables. The solution, however, normally involves infinite series, which are inconvenient and time-consuming to evaluate. Therefore, there is clear motivation to present the solution in tabular or graphical form. However, the solution involves the parameters x, L, t, k, α, h, Ti and T_∞ which are too many to make any graphical presentation of the results practical. In order to reduce the number of parameters, the problem can be nondimensionalized by defining the following dimensionless quantities;

Dimensionless temperature $\theta = \frac{T - T_\infty}{T_i - T_\infty}$ (2.1)

Dimensionless distance $\bar{x} = \frac{x}{L}$ (2.2)

Dimensionless time $\tau = \frac{\alpha t}{L^2}$ (2.3)

Dimensionless heat transfer coefficient $Bi = \frac{hL}{k}$ (2.4)

The nondimensionalization enables us to present the temperature in terms of three parameters only: $\bar{x}$, Bi and τ. This makes it practical to present the solution in graphical form. The dimensionless quantities defined above for a plane wall can also be used for a cylinder or sphere by replacing the space variable x by r and the half thickness L by the outer radius r_o. It has been noted that the characteristic length in the definition of the Biot number is taken to be the half thickness L for the plane wall and the radius r_o for the long cylinder and sphere. [12]

2.2 Initial and Boundary Conditions

2.2.1 Initial Condition

It has been considered the plane wall. The temperature at any point on the wall at a specified time also depends on the condition of the wall at the beginning of the heat conduction process. Such a condition, which is usually specified at time $t = 0$, is called the initial condition, which is a mathematical expression for the temperature distribution of the medium initially. In rectangular coordinates, the initial condition for one dimensional can be specified in the general form as

$$t = 0 \qquad T = f(x) \tag{2.5}$$

Where the function f (x) represents the temperature distribution throughout the medium at time $t = 0$. When the medium is initially at a uniform temperature of T_i, the initial condition can be written as,

$$t = 0 \qquad T = T_i \tag{2.6}$$

2.2.2 Boundary Conditions

The heat conduction equations are obtained using an energy balance on a differential element inside the medium and they remain the same regardless of the thermal conditions on the surfaces of the medium. The mathematical expressions of the thermal conditions at the boundaries are called the boundary conditions. From a mathematical point of view, solving a differential equation is essentially a process of removing derivatives or an integration process and thus the solution of a differential equation typically involves arbitrary constants. It follows that to obtain a unique solution to a problem, there is need to specify more than just the governing differential equation. The boundary conditions specify at the boundary of the region under consideration. For convenience in the analysis the boundary conditions can be separated into the following categories.

2.2.2.1 Specified Temperature Boundary Condition

Specified temperature boundary conditions can be expressed as

$$T(0,t) = T_1 \qquad T(L,t) = T_2 \tag{2.7}$$

Where T_1 and T_2 are the specified temperatures at surfaces at $x = 0$ and x = L, respectively. The specified temperatures can be constant.

2.2.2.2 Specified Heat Flux (Symmetry) Boundary Condition

Some heat transfer problems possess thermal symmetry as a result of the symmetry in imposed thermal conditions. The temperature distribution in one half of the wall is the same as that in the other half. That is, the heat transfer problem in this wall has been possessed thermal symmetry about the centre plane at x = 0. Also, the direction of heat flow at any point in the wall has been toward the surface closer to the point, and there is no heat flow across the centre plane. Therefore, the centre plane can be seen as an insulated surface, and the thermal condition at this plane of symmetry can be expressed as

$$\frac{\partial T}{\partial x} = 0 \tag{2.8}$$

This result can also be deduced from a plot of temperature distribution with a maximum, and thus zero slopes, at the centre plane wall. In the case of cylindrical (or spherical) bodies having thermal symmetry about the centre line (or midpoint), the thermal symmetry boundary condition requires that the first derivative of temperature with respect to (the radial variable) be zero at the centreline (or the midpoint).

2.2.2.3 Convection Boundary Condition

Convection is probably the most common boundary condition encountered in practice since most heat transfer surfaces are exposed to an environment at a specified temperature. The convection boundary condition is based on a surface energy balance expressed as

$$\begin{bmatrix} \text{Heat conduction at the} \\ \text{surface in} \\ \text{a selected direction} \end{bmatrix} = \begin{bmatrix} heat\ convection\ at\ the \\ surfaces\ \ in\ the \\ same\ direction \end{bmatrix}$$

For one dimensional heat transfer in the x direction in a plate of thickness 2L, the convection boundary conditions one surfaces can be expressed as

$$-k\frac{\partial T}{\partial x} = h\,(T - T_\infty) \tag{2.9}$$

Where h *is* the convection heat transfer coefficients and T_∞ is the temperatures of the surrounding medium. Convection boundary condition becomes as follows.

$$\frac{\partial T}{\partial x} + H(T - T_\infty) = 0 \tag{2.10}$$

Where $H = \frac{h}{k}$ is assumed as a constant. [13]

2.3 Analytical Solution

2.3.1 Large Plane Wall

The transient temperature distribution in relatively thin slab of material, which is cooled one or both side, can be approximated, away from the edge, as one dimensional. Such situations are frequently encountered in engineering application. As an example, a plane wall has been considered, initially at t = 0 at a uniform temperature T_i, which is subjected to the same cooling conditions on both sides for times $t \geq 0$. This plane wall is considered to be sufficiently large in the y and z directions compared to its thickness 2L in the x direction.

The heat transfer coefficient *h* on both surfaces is assumed to be constant. It is further assumed that the surrounding fluid temperature remains constant at T_∞ during to whole cooling process. Under these conditions heat flow through the plane wall has been one-dimensional in the x direction. If the thermophysical properties (k, ρ and c) are also assumed to be constant, the temperature differential equation is obtained as follows

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.11}$$

With the initial condition

$$T(x, 0) = T_i \tag{2.12}$$

And the boundary conditions

$$x = 0 \qquad \frac{\partial T}{\partial x} = 0 \tag{2.13}$$

$$x = L \qquad \frac{\partial T}{\partial x} + H(T - T_\infty) = 0 \tag{2.14}$$

2.3.1.1 Nondimensionlizing Differential Equation

Transient heat conduction differential equation for plane wall has been expressed as

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Various dimensionless quantities are defined as follows

Dimensionless distance $\bar{x} = \frac{x}{L}$ (2.15)

Dimensionless temperature $\theta = \frac{T - T_\infty}{T_i - T_\infty}$ (2.16)

Dimensionless time $\tau = \frac{\alpha t}{L^2}$ (2.17)

The left hand side of equation (2.11)

$$\bar{x} = \frac{x}{L}$$

$$\frac{\partial \bar{x}}{\partial x} = \frac{1}{L}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x}$$

$$\frac{\partial}{\partial x} = \frac{1}{L} \frac{\partial}{\partial \bar{x}}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{L}\frac{\partial}{\partial \bar{x}}\right) = \left(\frac{1}{L}\frac{\partial}{\partial \bar{x}}\right)\left(\frac{1}{L}\frac{\partial}{\partial \bar{x}}\right) = \frac{1}{L^2}\frac{\partial^2}{\partial \bar{x}^2}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{L^2}\frac{\partial^2 T}{\partial \bar{x}^2}$$

$$\theta = \frac{T - T_\infty}{T_i - T_\infty}$$

$$T = \theta\,(T_i - T_\infty) + T_\infty$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{L^2}\frac{\partial}{\partial \bar{x}}\left(\frac{\partial\,[\theta\,(T_i - T_\infty) + T_\infty]}{\partial \bar{x}}\right) = \frac{1}{L^2}\frac{\partial}{\partial \bar{x}}\left((T_i - T_\infty)\ \frac{\partial \theta}{\partial \bar{x}}\right)$$

The left hand side of eqn. (2.11) is defined as follows

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{L^2}\,(T_i - T_\infty)\,\frac{\partial^2 \theta}{\partial \bar{x}^2} \qquad (2.18)$$

The right hand side of eqn. (2.11)

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \tau}\frac{\partial \tau}{\partial t}$$

where $\tau = \frac{\alpha t}{L^2}$

then $\frac{\partial \tau}{\partial t} = \frac{\alpha}{L^2}$

$$\frac{\partial T}{\partial t} = \frac{\alpha}{L^2}\frac{\partial T}{\partial \tau}$$

$$T = \theta\,(T_i - T_\infty) + T_\infty$$

$$\frac{\partial T}{\partial t} = \frac{\alpha}{L^2}\frac{\partial[\theta(T_i - T_\infty) + T_\infty]}{\partial \tau}$$

The right hand side of eqn. (2.11) are defined as follows

$$\frac{\partial T}{\partial t} = \frac{\alpha}{L^2}(T_i - T_\infty)\frac{\partial \theta}{\partial \tau} \tag{2.19}$$

Substituting eqn. (2.18) and eqn.(2.19) in eqn. (2.11), yields

$$\frac{1}{L^2}\,(T_i - T_\infty)\,\frac{\partial^2 \theta}{\partial \bar{x}^2} = \frac{1}{\alpha}\,\frac{\alpha}{L^2}(T_i - T_\infty)\frac{\partial \theta}{\partial \tau}$$

So the correlation of energy equation has been obtained as

$$\frac{\partial^2 \theta}{\partial \bar{x}^2} = \frac{\partial \theta}{\partial \tau} \tag{2.20}$$

2.3.1.2 Analytical Solution with Separation of Variables

Correlation of energy equation has been defined as

$$\frac{\partial^2 \theta}{\partial \bar{x}^2} = \frac{\partial \theta}{\partial \tau}$$

With the initial condition

$$\tau = 0 \quad \theta = 1 \tag{2.21}$$

and the boundary conditions

$$\bar{x} = 0 \qquad \frac{\partial \theta}{\partial \bar{x}} = 0 \tag{2.22}$$

$$\bar{x} = 1 \qquad \frac{\partial \theta}{\partial \bar{x}} + Bi\theta = 0 \tag{2.23}$$

The dimensionless temperature θ is assumed as a product of X *and* Γ where X is function of $\bar{x}$ only and Γ is the function of τ only.

$$\theta = X\Gamma$$

$$X = f(\bar{x}) \quad \Gamma = f(\tau)$$

$$\frac{1}{X}\frac{d^2X}{d\bar{x}^2} = \frac{1}{\Gamma}\frac{d\Gamma}{d\tau} = -\lambda^2$$

In this equation, λ is a constant. The left hand side is a function of the space variable $\bar{x}$, only, and the right hand side of the time variable τ, alone; the only way this equality holds if both sides are equal to the same constant. And problem has been become as

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\tau} = -\lambda^2 \tag{2.24}$$

$$\frac{d^2X}{d\bar{x}^2} + \lambda^2 X = 0 \tag{2.25}$$

Solving of eqn. (2.24), yields

$$\Gamma = Ce^{-\lambda^2\tau} \tag{2.26}$$

Solving of eqn. (2.25),

$$X = A\sin\lambda\bar{x} + B\cos\lambda\bar{x} \tag{2.27}$$

The boundary conditions for $X = f(\bar{x})$

$$\bar{x} = 0 \qquad \frac{\partial\theta}{\partial x} = 0$$

$$\theta = X\Gamma \qquad \text{then} \qquad \frac{\partial\theta}{\partial\bar{x}} = \Gamma\frac{dX}{d\bar{x}} = 0 \quad \frac{dX}{d\bar{x}} = 0$$

$$\bar{x} = 1 \qquad \frac{\partial\theta}{\partial\bar{x}} + Bi\theta = 0$$

$$\Gamma\frac{dX}{d\bar{x}} + \Gamma BiX = 0 \qquad \text{then} \qquad \Gamma\left[\frac{dX}{d\bar{x}} + BiX\right] = 0$$

$$\bar{x} = 0 \qquad \frac{dX}{d\bar{x}} = 0 \tag{2.28}$$

$$\bar{x} = 1 \qquad \frac{dX}{d\bar{x}} + BiX = 0 \tag{2.29}$$

Applying the boundary condition eqn.(2.28) for;

$$X = A\sin\lambda\,\bar{x} + B\cos\lambda\bar{x} = 0$$

$$\bar{x} = 0 \quad \frac{dX}{d\bar{x}} = A\,\lambda\cos\lambda\,\bar{x} - B\,\lambda\sin\lambda\bar{x} = 0$$

$$= A\lambda\cos\lambda\,(0) - B\lambda\sin\lambda(0) = 0$$

$$A\lambda = 0 \qquad A = 0$$

then $X = B\cos(\lambda\bar{x})$ (2.30)

Applying the boundary condition eqn. (2.29) for;

$$\bar{x} = 1$$

$$-\lambda B\sin(\lambda\bar{x}) + BiB\cos(\lambda\bar{x}) = 0$$

$$-\lambda B\sin(\lambda) + BiB\cos(\lambda) = 0$$

$$Bi = \lambda\tan\lambda \tag{2.31}$$

Separation of variables $\theta\,(\bar{x},\tau) = X(\bar{x})\Gamma(\tau)$ [13]

$$\theta = X\Gamma = B\cos(\lambda_n\bar{x})\,Ce^{-\lambda_n{}^2\tau}$$

$$BC = A_n$$

$$\theta = \sum_{n=1}^{\infty} A_n\cos(\lambda_n\bar{x})\,e^{-\lambda_n{}^2\tau} \tag{2.32}$$

The constant A_n in eqn. (2.32) are determined from the application of the initial condition eqn. (2.21) that is

$$\tau = 0 \qquad \theta = 1$$

$$1 = \sum_{n}^{\infty} A_n\cos(\lambda_n\bar{x}) \qquad \text{where} \qquad A_n = \frac{\int_{-1}^{1}\cos(\lambda_n\bar{x})d\bar{x}}{N}$$

$$N = \int_{-1}^{1} \cos^2(\lambda_n \bar{x})\, d\bar{x}$$

It can be rewritten as

$$N = \int_{-1}^{0} \cos^2(\lambda_n \bar{x}) d\bar{x} + \int_{0}^{1} \cos^2(\lambda_n \bar{x}) d\bar{x}$$

$$= \frac{\bar{x}}{2} + \frac{2\sin\lambda_n\bar{x} + \cos\lambda_n\bar{x}}{4\lambda_n} \Big|_{-1}^{0} + \frac{\bar{x}}{2} + \frac{2\sin\lambda_n\bar{x} + \cos\lambda_n\bar{x}}{4\lambda_n} \Big|_{0}^{1}$$

$$= -\left[-\frac{1}{2} + \frac{2\sin(-\lambda_n) + \cos(-\lambda_n)}{4\lambda_n}\right] + \left[\frac{1}{2} + \frac{2\sin(-\lambda_n) + \cos(-\lambda_n)}{4\lambda_n}\right]$$

$$= +\left[\frac{1}{2} + \frac{2\sin(-\lambda_n) + \cos(-\lambda_n)}{4\lambda_n}\right]$$

$$= \frac{1}{2} + \frac{2\sin\lambda_n + \cos\lambda_n}{4\lambda_n} + \frac{1}{2} + \frac{2\sin\lambda_n + \cos\lambda_n}{4\lambda_n}$$

$$N = 1 + \frac{\sin\lambda_n \cos\lambda_n}{\lambda_n}$$

$$\int_{-1}^{1} \cos(\lambda_n \bar{x}) dx = \frac{1}{\lambda_n} \sin\lambda_n(\bar{x}) \Big|_{-1}^{1} = \frac{1}{\lambda_n}[\sin\lambda_n - \sin(-\lambda_n)] = 2\frac{\sin\lambda_n}{\lambda_n}$$

$$A_n = \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)} \tag{2.33}$$

Substituting A_n in eqn. (2.32), dimensionless temperature distribution has been defined as follows,

$$\theta = \sum_{n=1}^{\infty} \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)} \cos(\lambda_n \bar{x})\, e^{-\lambda^2\tau} \tag{2.34}$$

Where the characteristic values λ_n are the positive roots of

$$Bi = \lambda_n \tan\lambda_n \tag{2.35}$$

Where $Bi = \frac{h}{k}$. Here, it is to be noted that the number of roots of eqn. (2.35) are infinite. Although they cannot be obtained by ordinary algebraic methods, they can be found numerically or determined graphically.

2.3.2 Long Cylinder

The solid cylinder has been let under consideration be heated initially to some known axially symmetric temperature distribution T and then suddenly at t = 0, be placed in contact with a fluid at temperature T_∞. The heat transfer coefficient h at the surface of the cylinder is constant. In addition, it has been assumed that the temperature of the surrounding fluid remains constant at T_∞ during the whole cooling process for $t \geq 0$. If the thermophysical properties (k, ρ and c) are also assumed to be constant, the temperature differential equation is defined as follows

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.36)$$

With the initial condition

$$T(r, 0) = T_i \qquad (2.37)$$

And the boundary conditions (symmetric)

$$r = 0 \qquad \frac{\partial T}{\partial r} = 0 \qquad (2.38)$$

$$r = r_0 \qquad \frac{\partial T}{\partial r} + H(T - T_\infty) = 0 \qquad (2.39)$$

2.3.2.1 Nondimensionlizing Differential Equation

Transient heat conduction differential equation for long cylinder has been expressed as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Various dimensionless quantities are defined as follows

$$\bar{r} = \frac{r}{r_0} \text{ = dimensionless distance} \qquad (2.40)$$

$$\theta = \frac{T - T_\infty}{T_i - T_\infty} \text{ = dimensionless temperature} \qquad (2.41)$$

$$\tau = \frac{\alpha t}{{r_0}^2} \text{ = dimensionless time (Fourier number)} \qquad (2.42)$$

Nondimesionlizing left hand side of eqn. (2.36) as

$$\bar{r} = \frac{r}{r_0} \qquad \frac{\partial \bar{r}}{\partial r} = \frac{1}{r_0}$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \bar{r}}\left(\frac{\partial \bar{r}}{\partial r}\right)$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \bar{r}}\left(\frac{1}{r_0}\right) \qquad \frac{\partial}{\partial r} = \left(\frac{1}{r_0}\right)\frac{\partial}{\partial \bar{r}}$$

$$\frac{\partial T}{\partial r} = \left(\frac{1}{r_0}\right)\frac{\partial T}{\partial \bar{r}}$$

$$\theta = \frac{T - T_\infty}{T_i - T_\infty}$$

$$T = \theta(T_i - T_\infty) + T_\infty$$

$$\frac{\partial T}{\partial \bar{r}} = \frac{\partial[\theta(T_i - T_\infty) + T_\infty]}{\partial \bar{r}}$$

$$\frac{\partial T}{\partial \bar{r}} = (T_i - T_\infty)\frac{\partial \theta}{\partial \bar{r}}$$

$$\frac{\partial T}{\partial \bar{r}} = \left(\frac{1}{r_0}\right)(T_i - T_\infty)\frac{\partial \theta}{\partial \bar{r}} \qquad \text{and} \qquad \bar{r} = \frac{r}{r_0} \qquad r = \bar{r}\, r_0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\bar{r}.r_0}\left(\frac{1}{r_0}\right)\frac{\partial}{\partial \bar{r}}\left[\bar{r}\, r_0\left(\frac{1}{r_0}\right)(T_i - T_\infty)\frac{\partial \theta}{\partial \bar{r}}\right]$$

$$= \frac{1}{\bar{r}.r_0}\left(\frac{1}{r_0}\right)\frac{\partial}{\partial \bar{r}}\left[\bar{r}\, r_0\left(\frac{1}{r_0}\right)(T_i - T_\infty)\frac{\partial \theta}{\partial \bar{r}}\right]$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = (T_i - T_\infty)\frac{1}{\bar{r}}\frac{1}{{r_0}^2}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \theta}{\partial \bar{r}}\right) \tag{2.43}$$

Nondimesionlizing right hand side of eqn. (2.36) as

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \tau}\,\frac{\partial \tau}{\partial t}$$

$$\tau = \frac{\alpha t}{r_0^2} \qquad \frac{\partial T}{\partial \tau} = \frac{\alpha}{r_0^2}$$

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r_0^2}\frac{\partial T}{\partial \tau}$$

$$\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial \tau}[\theta(T_i - T_\infty) + T_\infty]$$

$$\frac{\partial T}{\partial \tau} = (T_i - T_\infty)\frac{\partial \theta}{\partial \tau}$$

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r_0^2}(T_i - T_\infty)\frac{\partial \theta}{\partial \tau}$$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r_0^2}(T_i - T_\infty)\frac{\partial \theta}{\partial \tau} \tag{2.44}$$

Substituting eqn. (2.43) and eqn. (2.44) in eqn. (2.36),

$$(T_i - T_\infty)\frac{1}{\bar{r}}\frac{1}{{r_0}^2}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\,\frac{\partial \theta}{\partial \bar{r}}\right) = \frac{1}{r_0^2}(T_i - T_\infty)\frac{\partial \theta}{\partial \tau}$$

Energy equation has been obtained as follows;

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\,\frac{\partial \theta}{\partial \bar{r}}\right) = \frac{\partial \theta}{\partial \tau} \tag{2.45}$$

For dimensionless temperature distribution, the initial condition

$$\tau = 0 \qquad \theta = 1 \tag{2.46}$$

And the boundary conditions

$$\bar{r} = 0 \qquad \frac{\partial \theta}{\partial \bar{r}} = 0 \tag{2.47}$$

$$\bar{r} = 1 \qquad \frac{\partial \theta}{\partial \bar{r}} + Bi\theta = 0 \tag{2.48}$$

2.3.2.2 Analytical Solution with Separation of Variables

Transient heat conduction correlation of energy equation for cylinder has been expressed as

$$\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\,\frac{\partial \theta}{\partial \bar{r}}\right) = \frac{\partial \theta}{\partial \tau}$$

The boundary conditions

$$\bar{r} = 0 \qquad \frac{\partial\theta}{\partial\bar{r}} = 0 \tag{2.49}$$

$$\bar{r} = 1 \qquad \frac{\partial\theta}{\partial\bar{r}} + Bi\theta = 0 \tag{2.50}$$

And the initial condition

$$\tau = 0 \qquad \theta = 1 \tag{2.51}$$

Problem solution with separation of variables [13]

$$\theta = R\Gamma \qquad R = f(\bar{r}) \quad \text{and} \quad \Gamma = f(\tau)$$

$$\frac{\partial\theta}{\partial\bar{r}} = \Gamma\frac{dR}{d\bar{r}}$$

$$\frac{\partial\theta}{\partial\tau} = R\frac{d\Gamma}{d\tau}$$

$$\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\Gamma\frac{dR}{d\bar{r}}\right) = R\frac{d\Gamma}{d\tau}$$

$$\frac{1}{R}\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\frac{dR}{d\bar{r}}\right) = \frac{1}{\Gamma}\frac{d\Gamma}{d\tau} = -\lambda^2$$

In this equation, λ is a constant. The left hand side is a function of the space variable $\bar{r}$, only, and the right hand side of the time variable τ, only; the only way this equality holds if both sides are equal to the same constant. And problem can be become as

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\tau} = -\lambda^2 \tag{2.52}$$

$$\frac{1}{R}\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\frac{dR}{d\bar{r}}\right) = -\lambda^2 \tag{2.53}$$

Solving of eqn. (2.52) is obtained as follows

$$\Gamma = Ce^{-\lambda^2\tau} \tag{2.54}$$

Eqn. (2.53) becomes as;

$$\frac{1}{R}\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\,\frac{dR}{d\bar{r}}\right) = -\lambda^2$$

$$\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\,\frac{dR}{d\bar{r}}\right) = -\lambda^2 R$$

$$\frac{1}{\bar{r}}\left[\frac{dR}{d\bar{r}} + \bar{r}\,\frac{d^2R}{d\bar{r}^2}\right] = -\lambda^2 R$$

Bessel function equation of order 0 ($\nu = 0$) [13]

$$\frac{d^2R}{d\bar{r}^2} + \frac{1}{\bar{r}}\frac{dR}{d\bar{r}} + \lambda^2 R = 0 \qquad (3.55)$$

Solving of eqn. (2.55)

$$R = AJ_0(\lambda\bar{r}) + BY_0(\lambda\bar{r}) \qquad (2.56)$$

The boundary condition for Bessel function equation of order 0

$$\bar{r} = 0 \qquad \frac{\partial\theta}{\partial\bar{r}} = 0$$

$$\frac{\partial}{\partial\bar{r}}(R\Gamma) = 0 \qquad \frac{dR}{d\bar{r}} = 0$$

$$\bar{r} = 0 \qquad \frac{dR}{d\bar{r}} = 0 \qquad (2.57)$$

$$\bar{r} = 1 \qquad \frac{\partial\theta}{\partial\bar{r}} + Bi\theta = 0$$

$\Gamma\frac{dR}{d\bar{r}} + BiR\Gamma = 0$ then $\Gamma\left[\frac{dR}{d\bar{r}} + BiR\right] = 0$

$$\bar{r} = 0 \qquad \frac{dR}{d\bar{r}} = 0$$

$$\bar{r} = 1 \qquad \frac{dR}{d\bar{r}} + BiR = 0 \qquad (2.58)$$

Applying first boundary condition,

$$\bar{r} = 0 \qquad \frac{dR}{d\bar{r}} = 0 \qquad \frac{dR}{d\bar{r}} = \left[A\frac{d}{d\bar{r}}J_0(\lambda\bar{r}) + B\frac{d}{d\bar{r}}Y_0(\lambda\bar{r})\right]_{\bar{r}=0} = 0$$

$$\frac{d}{d\bar{r}} J_0(\lambda \bar{r}) = -\lambda J_1(\lambda \bar{r})$$

$$\frac{d}{d\bar{r}} Y_0(\lambda \bar{r}) = -\lambda Y_1(\lambda \bar{r})$$

$$[-A\lambda J_1(\lambda \bar{r}) - B\lambda Y_1(\lambda \bar{r})]_{\bar{r}=0} = 0$$

$-A\lambda J_1(0) - B\lambda Y_1(0) = 0$ $\quad J_1(0) = 0$ and $Y_1(0) = -\infty$ $\quad B = 0$

$$R = AJ_0(\lambda \bar{r}) \tag{2.59}$$

Applying second boundary condition

$\bar{r} = 1$ $\quad R = AJ_0(\lambda \bar{r})$ $\quad \frac{dR}{d\bar{r}} = \frac{d}{d\bar{r}}[AJ_0(\lambda \bar{r})] = -A\lambda J_1(\lambda \bar{r})$

$$\frac{dR}{d\bar{r}} + BiR = 0$$

$$[-A\lambda J_1(\lambda \bar{r}) + BiAJ_0(\lambda \bar{r})]_{\bar{r}=1} = 0$$

$$A[-\lambda J_1(\lambda \bar{r}) + BiJ_0(\lambda \bar{r})]_{\bar{r}=1} = 0$$

$$\lambda J_1(\lambda) = BiJ_0(\lambda)$$

$$Bi = \frac{\lambda J_1(\lambda)}{J_0(\lambda)}$$

Biot number $\text{Bi} = \lambda_n \frac{J_1(\lambda_n)}{J_0(\lambda_n)}$ (2.60)

Separation of variables as

$$\theta\,(\bar{r}, \tau) = X(\bar{r})\Gamma(\tau)$$

$$\theta = \sum_{n=1}^{\infty} A_n J_0(\lambda_n \bar{r}) e^{-\lambda^2 \tau} \tag{2.61}$$

Applying the initial condition eqn. (2.51), it can be obtained as

$\tau = 0 \qquad \theta = 1$

$$1 = \sum_{n=1}^{\infty} A_n J_0(\lambda_n \bar{r}) 1$$

Which a Fourier – Bessel expansion of θ with the values is of λ_n obtained from eqn. (2.61) hence, it has been

$$A_n = \frac{\int_0^1 \bar{r} J_0(\lambda_n \bar{r}) d\bar{r}}{\int_0^1 \bar{r} J^2{}_0(\lambda_n \bar{r}) d\bar{r}}$$

$$\int_0^1 \bar{r} J_0(\lambda_n \bar{r}) d\bar{r} = \frac{J_1(\lambda_n)}{\lambda_n}$$

$$N = \int_0^1 \bar{r} J^2{}_0(\lambda_n \bar{r}) d\bar{r} = \frac{2}{J^2{}_0(\lambda_n)} \frac{\lambda^2{}_n}{\lambda^2{}_n\left(\frac{J^2{}_1(\lambda_n)}{J^2{}_0(\lambda_n)}+1\right)}$$

$$A_n = \frac{2}{\lambda_n} \frac{J_1(\lambda_n)}{J^2{}_0(\lambda_n)+J^2{}_1(\lambda_n)} \quad (2.62)$$

Substituting A_n in eqn.(2.61), Thus, the solution for the temperature distribution becomes

$$\theta = \sum_{n=1}^{\infty} \frac{2}{\lambda_n} \frac{J_1(\lambda_n)}{J^2{}_0(\lambda_n)+J^2{}_1(\lambda_n)} J_0(\lambda_n \bar{r})\, e^{-\lambda^2 \tau} \quad (2.63)$$

Where the characteristic values λ_n are the positive roots of

$$\text{Bi} = \lambda_n \frac{J_1(\lambda_n)}{J_0(\lambda_n)} \quad (2.64)$$

Where $Bi = \frac{h}{k}$. Here, it is to be noted that the number of roots of eqn.(2.64) are infinite. Although they cannot be obtained by ordinary algebraic methods, they can be found graphically.

2.3.3 Solid Sphere

For heat conduction problems posed in spherical coordinate systems which involve spherical symmetry. The transient heat conduction, in the absence of internal energy generation and if the thermophysical properties (k, ρ and c) are also assumed to be constant, the temperature differential equation is defined as follows

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.65}$$

It has been considered the cooling of a solid sphere of radius r_0, for times t ≥ 0, in a medium maintained at a constant temperature T_∞ with a constant heat transfer coefficient h. It has been assumed that the sphere under consideration has an initial temperature differential equation given by T_i at t = 0.

The initial condition

$$t = 0 \qquad T(r, 0) = T_i \tag{2.66}$$

And the boundary conditions

$$r = 0 \qquad \frac{\partial T}{\partial r} = 0 \tag{2.67}$$

$$r = r_0 \qquad \frac{\partial T}{\partial r} + H(T - T_\infty) = 0 \tag{2.68}$$

2.3.3.1 Nondimensionlizing Differential Equation

Transient heat conduction differential equation for sphere can be expressed as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Various dimensionless quantities are defined as follows

$$\bar{r} = \frac{r}{r_0} \text{ = dimensionless distance} \tag{2.69}$$

$$\theta = \frac{T - T_\infty}{T_i - T_\infty} \text{ = dimensionless temperature} \tag{2.70}$$

$$\tau = \frac{\alpha t}{{r_0}^2} \text{ = dimensionless time} \tag{2.71}$$

For nondimesionlizing left hand hand side of eqn. (2.65)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad r=\bar{r}\,r_0 \qquad \bar{r}=\frac{r}{r_0}$$

$$\frac{\partial}{\partial r}=\frac{\partial}{\partial \bar{r}}\frac{\partial \bar{r}}{\partial r}$$

$$\frac{d\bar{r}}{dr}=\frac{1}{r_0}$$

$$\frac{\partial T}{\partial t}=\frac{1}{r_0}\frac{\partial T}{\partial \bar{r}}$$

$$T=\theta(T_i-T_\infty)T_\infty$$

$$\frac{\partial T}{\partial r}=\frac{1}{r_0}(T_i-T_\infty)\frac{\partial T}{\partial \bar{r}} \qquad \frac{\partial T}{\partial \bar{r}}=\frac{\partial}{\partial \bar{r}}[\theta(T_i-T_\infty)]$$

$$\frac{1}{r^2}\frac{1}{r_0}\frac{\partial}{\partial \bar{r}}\left[r^2\frac{1}{r_0}(T_i-T_\infty)\frac{\partial T}{\partial \bar{r}}\right]$$

$$(T_i-T_\infty)\frac{1}{\bar{r}^2{r_0}^2}\frac{1}{r_0}\frac{\partial}{\partial \bar{r}}\left[\bar{r}^2{r_0}^2\frac{1}{r_0}\frac{\partial \theta}{\partial \bar{r}}\right]$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=(T_i-T_\infty)\frac{1}{\bar{r}^2}\frac{1}{{r_0}^2}\frac{\partial}{\partial \bar{r}}\left[\bar{r}^2\frac{\partial \theta}{\partial \bar{r}}\right] \tag{2.72}$$

For nondimesionlizing right hand side of eqn. (2.65)

$$\tau=\frac{\alpha t}{{r_0}^2} \qquad \frac{\partial}{\partial t}=\frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} \qquad \frac{\partial \tau}{\partial t}=\frac{\alpha}{{r_0}^2}$$

$$\frac{\partial \tau}{\partial t}=\frac{\alpha}{{r_0}^2}\frac{\partial T}{\partial \tau}$$

$$\frac{1}{\alpha}\frac{\partial \tau}{\partial t}=\frac{1}{{r_0}^2}\frac{\partial T}{\partial \tau} \qquad \frac{\partial T}{\partial \tau}=\frac{\partial}{\partial \tau}[\theta(T_i-T_\infty)+T_\infty]$$

$$\frac{\partial T}{\partial \tau}=(T_i-T_\infty)\frac{\partial \theta}{\partial \tau}$$

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{1}{{r_0}^2}\,(T_i-T_\infty)\frac{\partial \theta}{\partial \tau} \tag{2.73}$$

Substituting eqn. (2.72) and eqn.(2.73) in eqn. (2.65)

$$(T_i - T_\infty)\frac{1}{\bar{r}^2}\frac{1}{r_0{}^2}\frac{\partial}{\partial \bar{r}}\left[\bar{r}^2\frac{\partial \theta}{\partial \bar{r}}\right] = \frac{1}{r_0{}^2}\ (T_i - T_\infty)\frac{\partial \theta}{\partial \tau}$$

$$\frac{1}{\bar{r}^2}\frac{\partial}{\partial \bar{r}}\left[\bar{r}^2\frac{\partial \theta}{\partial \bar{r}}\right] = \frac{\partial \theta}{\partial \tau}$$

Governing equation of the solid sphere has been obtained as follows;

$$\frac{\partial^2 \theta}{\partial \bar{r}^2} + \frac{2}{r}\frac{\partial \theta}{\partial \bar{r}} = \frac{\partial \theta}{\partial \tau} \tag{2.74}$$

With the initial condition

$$\tau = 0 \quad \theta = 1 \tag{2.75}$$

and the boundary conditions

$$\bar{r} = 0 \qquad \theta = finite \tag{2.76}$$

$$\bar{r} = 1 \qquad \frac{\partial \theta}{\partial \bar{r}} + Bi\theta = 0 \tag{2.77}$$

2.3.3.2 Analytical Solution with Separation of Variables

Correlation of energy equation has been defined as follows

$$\frac{\partial^2 \theta}{\partial \bar{r}^2} + \frac{2}{r}\frac{\partial \theta}{\partial \bar{r}} = \frac{\partial \theta}{\partial \tau}$$

The boundary conditions

$$\bar{r} = 0 \qquad \theta = finite \tag{2.78}$$

$$\bar{r} = 1 \qquad \frac{\partial \theta}{\partial \bar{r}} + Bi\theta = 0 \tag{2.79}$$

With the initial condition

$$\tau = 0 \quad \theta = 1 \tag{2.80}$$

In terms of $\mathrm{U}(\bar{r},\tau) = \bar{r}\theta(\bar{r},\tau)$ [14], this problem has been become as follows

$$\frac{\partial U}{\partial \bar{r}^2} = \frac{\partial U}{\partial \tau}$$

The boundary conditions

$$\bar{r} = 0 \qquad\qquad U = 0 \tag{2.81}$$

$$\bar{r} = 1 \qquad\qquad \frac{\partial U}{\partial r} = \left[1 - \frac{h}{k}\right] U \tag{2.82}$$

Since the boundary conditions eqn.(2.81) and eqn.(2.82) are homogeneous the assumption of the existence of product solution of the form $\mathrm{U}(\bar{r}, \tau) = R(\bar{r})\, \Gamma(\tau)$ yields

$$\frac{\partial^2 \mathrm{U}}{\partial \bar{r}^2} = \frac{\partial U}{\partial \tau} \qquad\qquad U = R\,\Gamma \qquad R = f(\bar{r}) \qquad \Gamma = f(\tau)$$

$$\frac{1}{R}\frac{d^2 R}{d\bar{r}^2} = \frac{1}{\Gamma}\frac{d\Gamma}{d\tau} = -\lambda^2$$

In this equation, λ is a constant. The left hand side is a function of the space variable $\bar{r}$, only, and the right hand side of the time variable τ, only; the only way this equality holds if both sides are equal to the same constant. And problem has been become as

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\tau} = -\lambda^2 \tag{2.83}$$

$$\frac{d^2 R}{d\bar{r}^2} + \lambda^2 R = 0 \tag{2.84}$$

Solving of eqn.(2.84)

$$\Gamma = Ce^{-\lambda^2 \tau} \tag{2.85}$$

Solving of eqn.(2.84)

$$R = A\sin(\lambda\bar{r}) + B\cos(\lambda\bar{r}) \tag{2.86}$$

The boundary conditions for $R = f(\bar{r})$

$$\bar{r} = 0 \qquad\qquad R = 0 \tag{2.87}$$

$$\bar{r} = 1 \qquad \frac{\partial R}{\partial r} = \left[1 - \frac{h}{k}\right] R \tag{2.88}$$

$$\bar{r} = 0 \qquad R = 0$$

$$R = A\sin(0) + B\cos(0) = 0$$

$$B = 0$$

$$R = A\sin(\lambda \bar{r}) \tag{2.89}$$

Applying second boundary condition (2.88)

$$\bar{r} = 1 \qquad \frac{\partial R}{\partial r} = \left[1 - \frac{h}{k}\right] R$$

$$\frac{\partial R}{\partial r} = A\,\lambda\cos(\lambda \bar{r})$$

$$A\,\lambda\cos(\lambda \bar{r}) = \left[1 - \frac{h}{k}\right] A\sin(\lambda \bar{r})$$

$$A\left[\lambda\cos(\lambda \bar{r}) - \left[1 - \frac{h}{k}\right]\sin(\lambda \bar{r})\right] = 0 \qquad \text{and} \qquad A \neq 0$$

$$\lambda\cos(\lambda \bar{r}) - \left[1 - \frac{h}{k}\right]\sin(\lambda \bar{r}) = 0$$

$$\lambda\frac{\cos(\lambda \bar{r})}{\sin(\lambda \bar{r})} = \left[1 - \frac{h}{k}\right]\frac{\sin(\lambda \bar{r})}{\sin(\lambda \bar{r})}$$

$$\lambda\cot\lambda = \left[1 - \frac{h}{k}\right]$$

Where the characteristic values λ_n are the positive roots of

$$1 - \lambda_n \cot\lambda_n = Bi \tag{2.90}$$

Which is a transcendental equation obtained from the application of the boundary condition.

The characteristic values eqn.(2.90) can be solved numerically to obtain the characteristic values λ_n.

The use of the initial condition eqn.(2.80) yields

$$\tau = 0 \qquad \theta = \frac{T_i - T_\infty}{T_i - T_\infty} = 1$$

$$U = \sum_{n=1}^{\infty} A_n\, e^{-\lambda^2 \tau} \sin(\lambda_n \bar{r})$$

$$\bar{r} = \sum_{n=1}^{\infty} A_n \sin(\, \lambda_n \bar{r})$$

Where the expansion coefficients A_n as,

$$A_n = \frac{\int_0^1 \bar{r} \sin(\lambda_n \bar{r}) d\bar{r}}{\int_0^1 \sin^2(\lambda_n \bar{r}) d\bar{r}} \tag{2.91}$$

Then, the transient dimensionless temperature distribution in the sphere has been becomes as follows

$$\theta = \sum_{n=1}^{\infty} \frac{4(\sin \lambda_n - \lambda_n \cos \lambda_n)}{2\lambda_n - \sin(2\lambda_n)}\, e^{-\lambda^2 \tau} \frac{\sin(\lambda_n \bar{r})}{(\lambda_n \bar{r})} \tag{2.92}$$